Lecture for M.Sc. Semester I/II (Mathematics)

Paper –III (Topology)

TOPIC : SOME THEOREMS ON CONNECTEDNESS

Theorem 1: If every two points of a subset E of a topological space X are contained in some connected subset of E, then E is a connected subset of X.

Theorem 2: A topological space (X, τ) is connected iff every non empty proper subset of X has a non-empty frontier.

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Disconnected Space Definition : A topological space (X , τ) is said to be a disconnected space if and only if –

1. X can be expressed as a union of two non empty τ - separated sets that is, iff there exists two subsets C and D of X such that
   X=C $\cup$ D , $C \neq \phi, D \neq \phi$ and $\overline{C} \cap D = \phi \,\&\, C \cap \overline{D} = \phi$
2. There exists a non-empty proper subset of X which is both τ -open and τ - closed.
3. X is the union of two non –empty disjoint τ -open sets .

OR

X is the union of two non-empty disjoint τ -closed sets.

Theorem required to prove theorem 1: Let (X , τ) be a topological space and let E be a connected subset of X such that $E \subset A \cup B$ where A and B are separated sets. Then $E \subset A$ or $E \subset B$ that is, E cannot intersect both A and B.

THEOREMS

Theorem 1 : If every two points of a subset E of a topological space X are contained in some connected subset of E, then E is a connected subset of X.

Proof : Let (X , τ) be a topological space and E be a subset of X such that every two points of E are contained in some connected subset of E.

To prove that : E is connected.

Suppose that E is not connected , that is, E is disconnected. Then there exists two non-empty subsets A and B of X such that $E = A \cup B$

$$\overline{A} \cap B = \phi \;\&\; A \cap \overline{B} = \phi$$

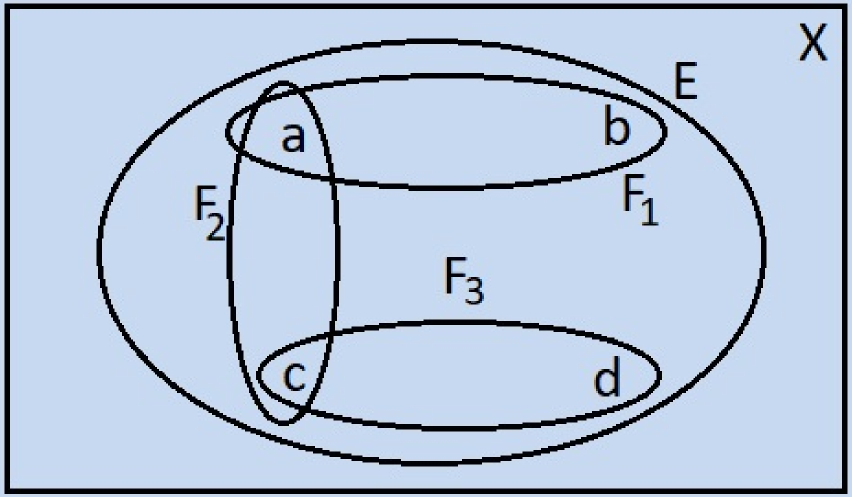
Since A and B are non-empty therefore there exists a point a in A and a point b in B. Since a and b are also the elements of E so by assumption there exists a connected subset F of E such that a, b are contained in F. Since $F \subset E \;\&\; E = A \cup B$, therefore $F \subset A \cup B$.

Using the theorem "If (X , τ) is a topological space and if E is a connected subset of X such that $E \subset A \cup B$ where A and B are separated sets. Then $E \subset A$ or $E \subset B$ that is, E cannot intersect both A and B."

We have $F \subset A$ or $F \subset B$. Since elements a and b both are in F, it implies that a , b both are either contained in A or contained in B. Suppose a, b be both are contained in A. Since b is also an element of B, from above discussion we have $A \cap B \neq \phi$ Thus we get a contradiction since A and B are disjoint sets.

Hence E must be a connected subset of X.

Hence proved.


X
E
a
b
F_1
F_2
F_3
c
d

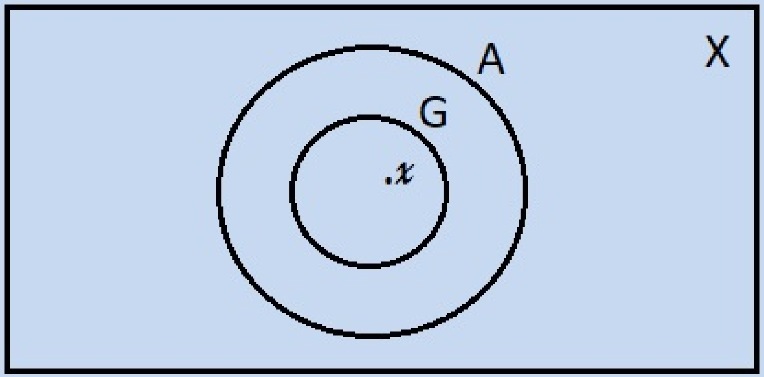
SOME DEFINITIONS

Interior points and the interior of a set:

Let (X, τ) be a topological space and let $A \subset X$. Then a point $x \in A$ is said to be an interior point of A iff A is a neighbourhood of x, that is , iff there exists an open set G such that

$x \in G \subset A$.

Also interior of A is the set of all interior points of A and is denoted by i(A) or Int(A) or A^o.



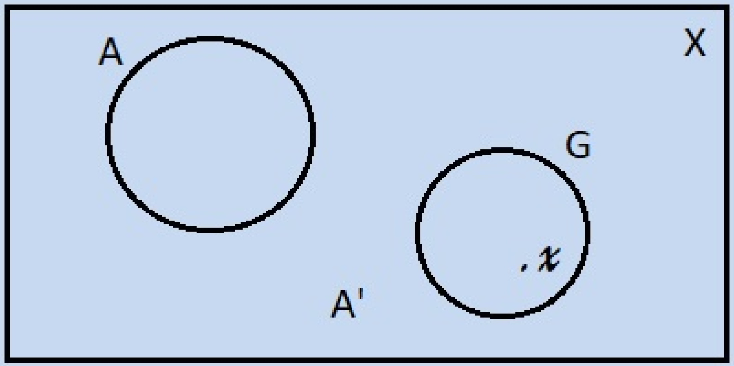
Exterior points and the exterior of a set :

Let (X, τ) be a topological space and let $A \subset X$ .Then a point x ϵ X is said to be an exterior point of A iff it is an interior point of A' , that is , iff there exists an open set G such that

$x \epsilon G \subset A'$.

Also exterior of A is the set of all exterior points of A and is denoted by ext(A) or e(A).

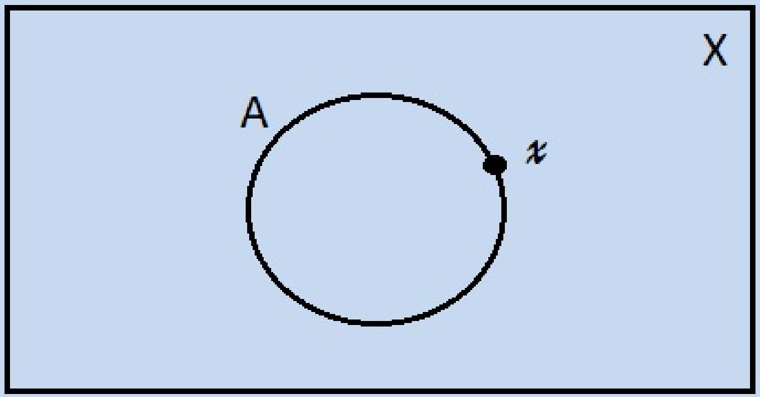
Thus ext(A) =Int(A') and Int(A) = ext(A').



Frontier points and the frontier of a set

Let (X , τ) be a topological space and let $A \subset X$. Then a point x ϵ X is said to be a frontier point (or boundary point) of A iff it is neither an interior point nor an exterior point of A.

Also frontier of A is the set of all frontier points of A and is denoted by Fr (A).



Points to remember:

1. $Fr(A) = \overline{A} - A^{o}$
2. $\overline{A} = A^{o} \cup Fr(A)$
3. $\overline{A} = A \cup Fr(\overline{A})$
4. A topological space (X, τ) is said to be a disconnected space iff there exists a non-empty proper subset of X which is both τ -open and τ - closed.

Theorem 2: A topological space (X , τ) is connected iff every non empty proper subset of X has a non-empty frontier.

Proof: Let (X , τ) be connected. We will prove that every non – empty proper subset of X has a non-empty frontier.

Suppose that there exists a subset A of X such that is A$\neq \phi$, A$\neq$X and Fr(A) = ϕ.

Therefore Fr($\overline{A}$)=ϕ (Because $\overline{A} \neq \phi$ & $\overline{A} \neq$ X)

We know that $\overline{A}$ = A^o $\cup$Fr(A) and $\overline{A}$ = A$\cup$ Fr($\overline{A}$)

Because Fr(A) = ϕ & Fr($\overline{A}$)=ϕ , therefore $\overline{A}$ = A^o = A.

$\overline{A}$ = A implies A is closed.

A^o = A implies A is open.

Thus A is a non-empty proper subset of X which is both open and closed implying that (X , τ) is disconnected. A contradiction to our assumption.

Hence every non-empty proper subset of X has a non-empty frontier.

Conversely, let (X , τ) be a topological space and A be a non-empty proper subset of X (that is A $\neq \phi$ and A $\neq$ X) such that Fr(A) $\neq \phi$ .We will prove that (X , τ) is connected.

Suppose that (X , τ) is disconnected. Then there exists a non-empty proper subset G of X which is both open and closed ,that is, $\overline{G}$ = G and G^o = G.

We know that Fr(G) = $\overline{G}$ - G^o which implies that Fr(G)=G – G = ϕ .
A contradiction to our assumption.

Hence (X , τ) is connected.

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THANKYOU